PMM U.S.S.R.,Vol.54,No.6,pp.812-815,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

ON THE CONVERSE LAGRANGE THEOREM IN MAGNETOHYDRODYNAMICS*

YU.G. GUBAREV

The problem of the converse Lagrange theorem /1, 2/ in magnetohydrodynamics is investigated. The linear problem of the stability of the quiescent state of a viscous incompressible fluid with an infinite conductivity containing a magnetic field is considered. It is shown by the direct Lyapunov method that the system is unstable if the second variation of the potential energy takes negative values. "A priori" lower and upper estimates of the increase in the perturbations are obtained. The lower estimate ensures an exponential increase in the displacements of the fluid particles and the lines of force of the magnetic field from the equilibrium state. The upper estimate shows that the solutions do not increase more rapidly than exponentially. In both cases, the exponents are calculated using the equilibrium state parameters and the initial data for the perturbation fields.

This paper extends the well-known results in /3, 4/ to magnetohydrodynamics.

1. Formulation of the exact problem. The spatial motions of a viscous, incompressible fluid with an infinite conductivity in a magnetic field are studied. The domain τ of the flow is bounded by a fixed, solid, ideally conducting boundary $\partial\tau$. The following notation is employed: p and $\mathbf{u} = (u_1, u_2, u_3)$ are the pressure and velocity fields, $\mathbf{h} = (h_1, h_2, h_3)$ is the magnetic field, $\mathbf{x} = (x_1, x_2, x_3)$ and t are Cartesian coordinates and the time, $\mathbf{n} = (n_1, n_2, n_3)$ is the normal to $\partial\tau$, ρ is the density of the fluid and η is the coefficient of dynamic viscosity. The equations of motion are taken in the form /5/

$$\begin{gathered}\rho du_i/dt = \sigma_{ik,k} + (4\pi)^{-1} h_k (h_{i,k} - h_{k,i}), \quad u_{k,k} = 0 \\ dh_i/dt = h_k u_{i,k}, \ h_{k,k} = 0 \\ (\sigma_{ik} = -p\delta_{ik} + \eta D_{ik}, \quad D_{ik} = u_{i,k} + u_{k,i}, \quad d/dt = \partial/\partial t + u_k \partial/\partial x_k \\ u_{i,k} = \partial u_i/\partial x_k, \ h_{i,k} = \partial h_i/\partial x_k, \ \sigma_{ik,k} = \partial\sigma_{ik}/\partial x_k)\end{gathered} \tag{1.1}$$

The conditions

$$u_i = 0, \quad h_i n_i = 0 \tag{1.2}$$

are satisfied on the boundary $\partial\tau$.

Everywhere, summation is carried out over repeated vector and tensor indices.

The energy dissipation equation

$$\begin{gathered}E_1^{\cdot} = -D_1; \ E_1 = K_1 + \Pi_1, \ D_1 = {}^1/_2 \int \eta D_{ik} D_{ik} d\tau \\ K_1 = {}^1/_2 \int \rho u_i u_i d\tau, \ \Pi_1 = (8\pi)^{-1} \int h_i h_i d\tau, \ d\tau = dx_1 dx_2 dx_3\end{gathered} \tag{1.3}$$

holds for problem (1.1) and (1.2).

The integration is carried out everywhere over the domain τ of the flow and a partial derivative with respect to time is indicated by a dot.

The exact stationary solutions of problem (1.1), (1.2)

$$\mathbf{u} = \mathbf{U}(\mathbf{x}) \equiv 0, \quad p = P(\mathbf{x}), \ \mathbf{h} = \mathbf{H}(\mathbf{x}) \tag{1.4}$$

which correspond to states of magnetostatic equilibrium, satisfy the equations

$$(4\pi)^{-1} H_k (H_{k,i} - H_{i,k}) = -P_{,i}, \ H_{k,k} = 0 \ (P_{,i} = \partial P/\partial x_i) \tag{1.5}$$

and the boundary conditions (1.2).

2. Formulation of the linearized problem. The linearization of problem (1.1), (1.2) on the solutions (1.4), taking account of (1.5), yields

*Prikl.Matem.Mekhan.,54,6,988-991,1990

$$\rho u_i'^{\cdot} = \sigma_{ik,k}' + (4\pi)^{-1}[h_k'(H_{i,k} - H_{k,i}) + H_k(h_{i,k}' - h_{k,i}')] \tag{2.1}$$
$$h_i'^{\cdot} = H_k u_{i,k}' - u_k' H_{i,k}, \quad u_{k,k}' = 0, \quad h_{k,k}' = 0 \quad \text{in} \quad \tau$$
$$u_i' = 0, \quad h_i' n_i = 0 \quad \text{on} \quad \partial\tau$$

Here, $\mathbf{u}'$, p' and $\mathbf{h}'$ are the perturbations, the velocity and pressure fields and the magnetic field. The expression for σ_{ik}' is identical with that for σ_{ik} when p and $\mathbf{u}$ are replaced by p' and $\mathbf{u}'$.

A field of Lagrangian displacements of the fluid particles $\boldsymbol{\xi}(\mathbf{x}, t)$ is introduced which satisfies the equation

$$\xi_i^{\cdot} = u_i' \tag{2.2}$$

The primes on the perturbation fields and on the tensors σ_{ik}' and D_{ik}' are omitted below.

In the case of the linearized problem (2.1), the energy dissipation equation has the form

$$E^{\cdot} = -D; \; E = K + \Pi, \; D = {}^1/_2 \int \eta D_{ik} D_{ik} d\tau \tag{2.3}$$
$$K = {}^1/_2 \int \rho u_i u_i d\tau, \; \Pi = (8\pi)^{-1} \int [h_i h_i - h_i \xi_k (H_{k,i} - H_{i,k})] \, d\tau$$

The functional Π (2.3) is identical to the expression for the second variation of the potential energy functional of problem (1.1), (1.2) /6/ since it is the first non-zero term in the expansion of Π_1 (1.3) close to the state of magnetostatic equilibrium (the quiescent state) (1.4), (1.5) and (1.2).

We will show that the quiescent state (1.4), (1.5), (1.2) is unstable when there is no minimum of the potential energy functional Π (2.3) and obtain estimates of the rate of increase in the perturbations.

It is assumed that a set of functions $\boldsymbol{\xi}(\mathbf{x})$ from (2.2) exists for which

$$\Pi < 0, \quad \boldsymbol{\xi}(\mathbf{x}) \in Q \tag{2.4}$$

In the case when $\boldsymbol{\xi}(\mathbf{x}) \notin Q$, the inequality (2.4) changes into the opposite inequality. Consequently, in the case of the functional Π, the quiescent state (1.4), (1.5) and (1.2) is an infinite dimensional analogue of a "saddle point".

3. ***The Lyapunov functional.*** The functionals /4/

$$M = \int \rho \xi_i \xi_i d\tau, \quad M^{\cdot} = 2 \int \rho \xi_i u_i d\tau \tag{3.1}$$
$$2G = \int \eta G_{ik} G_{ik} d\tau, \quad G_{ik} = \xi_{i,k} + \xi_{k,i}, \quad X = M^{\cdot} + G$$

are introduced.

Differentiation of the functional X with respect to time and subsequent reduction using (2.1), (2.3) and (3.1) yield the relationship

$$X^{\cdot} = 4(K - \Pi) = 8K - 4E \tag{3.2}$$

which is called the generalized virial equation /4/. By multiplying relationship (3.2) by an arbitrary constant factor, $-\lambda$, and adding it to the energy dissipation Eq.(2.3), it is possible to obtain the relationship

$$E_\lambda^{\cdot} = 2\lambda E_\lambda - 4\lambda K_\lambda - D_\lambda \tag{3.3}$$
$$E_\lambda = K_\lambda + \Pi_\lambda, \; 2\Pi_\lambda = 2\Pi + \lambda G + \lambda^2 M$$
$$2K_\lambda = 2K - \lambda M^{\cdot} + \lambda^2 M = \int \rho (\xi_t - \lambda \xi)^2 \, d\tau$$
$$D_\lambda = D - \lambda G^{\cdot} + \lambda^2 G = {}^1/_2 \int \eta (D_{ik} - \lambda G_{ik})^2 \, d\tau$$

Let $\lambda > 0$. Then, since the quantities K_λ and D_λ are non-negative, the inequality $E_\lambda^{\cdot} \leqslant 2\lambda E_\lambda$ follows from (3.3) and, on integrating this expression, we get

$$E_\lambda(t) \leqslant E_\lambda^\circ \exp(2\lambda t) \; (E_\lambda^\circ = E_\lambda(0)) \tag{3.4}$$

which holds for any solution of problem (2.1). It is important that no constraints whatsoever should be imposed on the sign of the potential energy functional Π (2.3) here. Since the functional E_λ varies monotonically, it may be treated as a Lyapunov functional.

4. ***The lower estimate.*** Let condition (2.4) be satisfied. This enables us to select the initial fields of the Lagrangian displacements $\boldsymbol{\xi}(\mathbf{x}, 0) \in Q$ such that $\Pi^\circ < 0$. The functions $\mathbf{u}(\mathbf{x}, 0)$, for which $K^\circ < |\Pi^\circ|$, are considered as the initial velocity fields.

The inequality

$$E^\circ < 0 \tag{4.1}$$

follows from the last two relationships.

By (3.3)

$$E_\lambda^\circ = E^\circ + \lambda A^\circ + \lambda^2 M^\circ, \quad 2A = G - M^\cdot \tag{4.2}$$

The functional E_λ° is a second degree polynomial in λ with a positive coefficient M° (3.1) accompanying λ^2 and a negative free term E° (4.1).

Let $\lambda > 0$. Then, in the interval,

$$0 < \lambda < \Lambda_1 = -{}^1/_2 A/M + \sqrt{({}^1/_2 A/M)^2 - E/M} \tag{4.3}$$

the relationship

$$E_\lambda^\circ < 0 \tag{4.4}$$

is satisfied.

The inequalities (3.4) and (4.4) show that the solutions of problem (2.1) increase exponentially with time.

If $\lambda = \Lambda_1 - \delta$ (for any δ from the interval $0 < \delta < \Lambda_1$), relationship (3.4) takes the form

$$E_{\Lambda_1-\delta}(t) \leqslant \overset{\circ}{E}_{\Lambda_1-\delta} \exp [2(\Lambda_1 - \delta) t] \; (\overset{\circ}{E}_{\Lambda_1-\delta} < 0) \tag{4.5}$$

The inequality

$$E_\lambda(t) = K_\lambda(t) + \Pi_\lambda(t) > \Pi(t)$$

follows from the definition of the functionals Π_λ and K_λ. This inequality, together with (4.5), yields the estimate

$$\Pi(t) < \overset{\circ}{E}_{\Lambda_1-\delta} \exp [2(\Lambda_1 - \delta) t] \tag{4.6}$$

Using the functional

$$J(t) = \int [h_i h_i + \xi_i \xi_i] \, d\tau$$

the inequality (4.6) is written in the more convenient form

$$J(t) > | c\overset{\circ}{E}_{\Lambda_1-\delta} | \exp [2(\Lambda_1 - \delta) t] \tag{4.7}$$

where c is a known constant.

It follows from (4.7) that the parameter Λ_1 (4.3) gives a lower estimate of the increments of the solution of problem (2.1).

The class of solutions of problem (2.1) is considered for which the initial velocity fields $\mathbf{u}(\mathbf{x}, 0)$ and the Lagrangian displacements $\xi(\mathbf{x}, 0)$ are associated at each point by the relationships

$$\mathbf{u}(\mathbf{x}, 0) = \lambda \xi(\mathbf{x}, 0) \tag{4.8}$$

It follows from relations (3.3) and (4.8) that

$$K_\lambda^\circ = 0, \quad E_\lambda^\circ = \Pi_\lambda^\circ \tag{4.9}$$

Let $\lambda > 0$ and let condition (2.4) be satisfied in the case of the Lagrangian displacement fields $\xi(\mathbf{x}, 0)$. Since, by (3.3),

$$2\Pi_\lambda^\circ = 2\Pi^\circ + \lambda G^\circ + \lambda^2 M^\circ$$

the inequality $\Pi_\lambda^\circ < 0$ holds in the interval

$$0 < \lambda < \Lambda = -{}^1/_2 G/M + \sqrt{({}^1/_2 G/M)^2 - 2\Pi/M} \tag{4.10}$$

By putting $\lambda = \Lambda - \delta$ (with an arbitrary δ from the interval $0 < \delta < \Lambda$) and taking account of relationship (4.9), inequality (3.4) can be written in the form

$$E_{\Lambda-\delta}(t) \leqslant \overset{\circ}{\Pi}_{\Lambda-\delta} \exp [2(\Lambda - \delta) t]$$

and the estimate

$$J(t) > | c\overset{\circ}{\Pi}_{\Lambda-\delta} | \exp [2(\Lambda - \delta) t] \tag{4.11}$$

follows from this.

Hence, the parameter Λ gives a lower estimate of the increments in the solutions of problem (2.1) from the class (4.8).

It will be shown below that the perturbations (4.8) are the most critical, since the most rapid growth in the solutions of problem (2.1) is observed when

$$\Lambda^+ = \sup_{\xi \in Q} \Lambda \tag{4.12}$$

5. *The upper estimate.* Let $\lambda > \Lambda^+$ (4.12). Then, the inequality

$$\Pi_\lambda > 0 \tag{5.1}$$

holds for the Lagrangian displacement fields $\xi(\mathbf{x}) \in Q$. By virtue of (2.4), relationship (5.1) is all the more satisfied in the case of the functions $\xi(\mathbf{x}) \notin Q$. Hence, the functional Π_λ is positive-definite for all possible Lagrangian displacement fields $\xi(\mathbf{x})$. Relations

(2.2), (3.3) and (5.1) show that the functional E_λ is also positive-definite for all possible Lagrangian displacement fields $\xi(x)$ and velocity fields $\mathbf{u}(x)$.

Consequently, the estimate

$$E_{\Lambda^++\varepsilon}(t) \leqslant \overset{\circ}{E}_{\Lambda^++\varepsilon} \exp[2(\Lambda^+ + \varepsilon)t]$$

follows from (3.4) when $\lambda = \Lambda^+ + \varepsilon$ $(\varepsilon > 0)$ which, by using the inequality $\Pi_{\Lambda^+}(t) \geqslant 0$, is transformed to the more obvious form

$$2K_{\Lambda^++\varepsilon}(t) + \varepsilon(2\Lambda^+ + \varepsilon)M(t) + \varepsilon G \leqslant 2\overset{\circ}{E}_{\Lambda^++\varepsilon} \exp[2(\Lambda^+ + \varepsilon)t] \qquad (5.2)$$

It is seen from (5.2) that the parameter $\Lambda^+ + \varepsilon$ gives an upper estimate of the increments in the solutions of problem (2.1). A comparison of estimates (4.11) and (5.2), taking account of (4.12), shows that the parameter Λ^+ gives both an upper and lower estimate of the rate of growth in the most critical perturbations (4.8):

$$\Lambda^+ - \delta \leqslant \omega_* \leqslant \Lambda^+ + \varepsilon$$

The author thanks V.A. Vladimirov for suggesting the problem and for useful discussions.

REFERENCES

1. CHATAYEV N.G., The Stability of Motion, Nauka, Moscow, 1965.
2. MOISEYEV N.N. and RUMYANTSEV V.V., Dynamics of a Body with Cavities containing a Fluid, Nauka, Moscow, 1965.
3. VLADIMIROV V.A. and RUMYANTSEV V.V., On the converse Lagrange theorem for a solid with a hollow containing an ideal fluid, Prikl. Matem. i Mekhan., 53, 4, 1989.
4. VLADIMIROV V.A. and RUMYANTSEV V.V., On the converse Lagrange theorem for a solid with a hollow containing a viscous fluid, Prikl. Matem. i Mekhan., 54, 2, 1990.
5. LANDAU L.D. and LIFSHITZ E.M., Theoretical Physics. 8, Electrodynamics of Continuous Media, Nauka, Moscow, 1982.
6. DEMUTSKII V.P. and POLOVIN R.V., Fundamentals of Magnetohydrodynamics, Energoatomizdat, Moscow, 1987.

Translated by E.L.S.

PMM U.S.S.R., Vol.54, No.6, pp.815-820, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

THE CONSTRUCTION OF RECIPROCITY AND INTEGRAL REPRESENTATION FORMULAS OF THE GENERAL SOLUTION FOR QUASISTATIC AND DYNAMIC PROBLEMS OF UNCOUPLED GENERALIZED THERMOELASTICITY*

YU.M. MAMEDOV

Reciprocity formulas are constructed and representations of the Somigliani-type are obtained for quasistatic and dynamic problems of uncoupled generalized thermoelasticity in the Lord-Shulman formulation that is effective for applications. Moreover, representations are obtained for the stresses and heat flux. Unlike the existing approach (/1/, say) these formulas are derived on the basis of an examination of the system of differential equations of the above-mentioned problems of generalized thermoelasticity as a system with appropriate non-selfadjoint differential operators. Operators adjoint to the initial differential operators are introduced into consideration for the construction of the reciprocity formulas (second Green's formula), and a Laplace transformation is used.

**Prikl.Matem.Mekhan.*, 54, 6, 992-997, 1990